

Extended Supersymmetry and Super-BF Gauge Theories*

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Abstract

The absence of fermion kinetic terms in supersymmetric- BF gauge theories is established. We do this by means of explicit off-shell (superspace) constructions. As part of our study we give the superspace constraints for $D=3$, $N=4$ super Yang-Mills along with the $D=3$, $N=4$ superconformal algebra. The puzzle we are interested in solving is the fact that the topological cousins, known as super- BF gauge theories, of certain supersymmetric- BF theories have kinetic terms for the twisted fermions. We show that the map which takes the latter to the former includes a Hodge decomposition of the twisted fermions. In conjunction with this result, we argue that it is natural to modify the naive path integral measure of supersymmetric- BF theories to include the Ray-Singer analytic torsion.

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1 Introduction

In this paper, we will construct and study various applications of extended supersymmetric BF gauge theories¹ (SUSY- BF) and D=3, N=4 supersymmetry in general. As one of the applications, we will show that a pair of problems associated with supersymmetric BF gauge theories (SUSY- BF) share a common solution in the form of an insertion of the Ray-Singer (R-S) torsion [2] in the measure of the path integral.

The first of these problems, as we will see, is the lack of fermionic determinants to cancel those from the bosons in these supersymmetric theories. Indeed, we will show that the fermionic contribution to the latter is only as an off-diagonal mass term. Although our explicit constructions will be primarily in three dimensions, we expect the results to hold for arbitrary dimensions.

The second problem arises as follows. It is well known that a large class of topological quantum field theories (TQFTs) may be obtained by twisting certain supersymmetric field theories. However, this procedure does not work for SUSY- BF theories. The supposed twisted cousins of these theories, which we shall call super- BF theories², have kinetic terms for the would-be twisted fermions. Thus far, super- BF theories have been constructed only via BRST gauge fixing.

To solve the first problem we will simply insert the Ray-Singer analytic torsion in the measure of the path integral for the SUSY- BF gauge theory. What is more interesting, we find that such an insertion is also needed in order to solve the second problem. In order to have a match between the twisted SUSY- BF and super- BF partition functions, one of the fermions obtained by twisting the spin- $\frac{1}{2}$ fields in the former theory must be Hodge decomposed. This change of variables results in the addition of the R-S torsion to the measure of the path integral³. This may be interpreted as defining a new vacuum for the SUSY- BF theory.

The paper is organized as follows. We begin in sections 2 (for D=3, N=1

¹For a review of non-supersymmetric BF theories and topological field theories (TFTs) in general, see reference [1].

²In order to avoid confusion in nomenclature we will refer to the ordinary supersymmetric BF theories as SUSY- BF while the corresponding TQFTs will be called super- BF theories.

³Recall that for closed manifolds, the R-S torsion is purely topological.

and $D=4, N=1$) and 3 (for $D=3, N=4$) by illustrating the presence of only (off-diagonal) mass terms for the fermions in SUSY- BF theories. This lack of kinetic terms for the fermions comes as no surprise due to two statements. First, supersymmetric Chern-Simons (CS) theories [3] are known to have only these mass terms due to the fact that the CS action give a mass for the gauge field. Second, BF theories with gauge group G may be obtained from CS theories [4] via the formation of an inhomogeneous gauge group out of G . Our motivation for constructing these theories is that we will later use them to illustrate the problem in twisting from SUSY- BF to super- BF theories. Additionally, our constructions of superspace geometries for $D=3, N=4$ super Yang-Mills will not only fill in a gap in the literature but will also serve to point out the richness of supermultiplets for the latter theories. Next, in section 4, we illustrate some novel features of SUSY- BF theories amongst which is the existence of a “minimal” $N=4$ action. Then in section 5, we twist the minimal $D=3, N=4$ SUSY- BF theory to obtain a topological gauge theory with a mass term for Grassmann-odd one- and two-form fields. We then show that in order to obtain the super- BF theory we must Hodge decompose the two-form field. Our conclusions may be found in section 6. As part of our general formulation of $D=3, N=4$ supersymmetric theories, we discuss the construction of off-shell $D=3, N=4$ supergravity and give the $D=3, N=4$ superconformal algebra in appendices A and B, respectively. Then, commensurate with our discussion of SUSY- BF theories, a proposed new action for $N=4$ $U(1)$ supersymmetric anyons is given in appendix C.

2 Generic Properties of SUSY- BF

In order to set the stage for our discussions let us begin by constructing, as exercises, the SUSY- BF $N=1$ supersymmetric actions in three and four space-time dimensions⁴.

⁴The notation $S_{SUSY}[D, N]$ is used to denote the (untwisted) D -dimensional, N -extended supersymmetric BF action.

2.1 D=3

N=1 superspace construction of the SUSY- BF action proceeds as follows. First, introduce a spin- $\frac{1}{2}$ superfield, \mathcal{B}_α , along with the spinor field-strength, W_α , of the super Yang-Mills theory [5, 3] and write the superfield action

$$S_{SUSY}[3, 1] = -\frac{1}{2} \int d^3\sigma d^2\theta \text{Tr}(\mathcal{B}^\alpha W_\alpha) . \quad (2.1)$$

Then with the component fields given by projection as

$$\begin{aligned} \mathcal{B}_\alpha| &= \beta_\alpha , & \nabla_\beta \mathcal{B}_\alpha| &= i(\gamma^a)_{\alpha\beta} B_a + C_{\beta\alpha} b , & \nabla^2 \mathcal{B}_\alpha| &= \rho_\alpha , \\ W_\alpha| &= \psi_\alpha , & \nabla_\beta W_\alpha| &= i\frac{1}{2}\epsilon^{abc}(\gamma_a)_{\alpha\beta} F_{bc} , \\ \nabla^2 W_\alpha| &= -i(\gamma^a)_{\alpha\beta} \mathcal{D}_a \psi^\beta , \end{aligned} \quad (2.2)$$

where \mathcal{D}_a is the Yang-Mills covariant derivative, we find the action to be

$$S_{SUSY}[3, 1] = \frac{1}{2} \int d^3\sigma \text{Tr}(\epsilon^{abc} B_a F_{bc} + i\beta^\alpha (\gamma^a)_{\alpha\beta} \mathcal{D}_a \psi^\beta - \rho^\alpha \psi_\alpha) . \quad (2.3)$$

The Dirac action in this expression is actually fictitious. This is seen as follows. Recall that the spinor field-strength of D=3, N=1 super Yang-Mills satisfies the Bianchi Identity, $\nabla^\alpha W_\alpha = 0$. This implies that the superspace action is invariant under the superfield transformation $\delta \mathcal{B}_\alpha = \nabla_\alpha \Lambda$. Writing, $\Lambda| = X$, $\nabla_\alpha \Lambda| = X_\alpha$ and $\nabla^2 \Lambda| = X'$, we see that the component transformations include $\delta \beta_\alpha = X_\alpha$, $\delta B_a = \mathcal{D}_a X$ and $\delta b = -X'$. Consequently, β_α and b may be set to zero algebraically. Hence, in this Wess-Zumino gauge, the component action reads

$$S_{SUSY}[3, 1] = \frac{1}{2} \int d^3\sigma \text{Tr}(\epsilon^{abc} B_a F_{bc} - \rho^\alpha \psi_\alpha) . \quad (2.4)$$

Observe that the fermions only appear in an off-diagonal mass term.

There is another feature of SUSY- BF theories which we would like to point out. Our (real) N=1 supersymmetric action is actually invariant under a complex N=1 supersymmetry transformation. This is best seen in superfield form for which the action is symmetric under the interchange of \mathcal{B}_α and W_α . Of course, these two superfields form representations of two completely different supermultiplets, thus we do not expect this symmetry

to be preserved (off-shell) at the level of the algebra. Nevertheless, we find that the component action is invariant under the complex supersymmetry transformations generated by Q_α and \bar{Q}_α for which

$$\begin{aligned}
[Q_\alpha, A_a] &= (\gamma_a)_\alpha{}^\beta (\psi_\beta + i\rho_\beta) \ , \\
\{Q_\alpha, \psi_\beta\} &= -\epsilon^{abc} (\gamma_a)_{\alpha\beta} (F_{bc} - i\mathcal{D}_{[b}B_{c]}) \ , \\
[Q_\alpha, B_b] &= i(\gamma_a)_\alpha{}^\beta (\psi_\beta - i\rho_\beta) \ , \\
\{Q_\alpha, \rho_\beta\} &= -i\epsilon^{abc} (\gamma_a)_{\alpha\beta} (F_{bc} + i\mathcal{D}_{[b}B_{c]}) \ .
\end{aligned} \tag{2.5}$$

Now this complex N=1 supersymmetry does not form a N=2 supersymmetry algebra off-shell. A quick check of this statement follows from computing $\{Q_\alpha, Q_\beta\}$ acting on B_a and finding that it is proportional to F_{ab} , the field strength of A_a . As $F_{ab} = 0$ is the equation which follows from varying the action with respect to B_a , we see that this anti-commutator vanishes only on-shell. In order to elevate this on-shell supersymmetry representation to that of an $N = 2$ off-shell supersymmetry we must include additional auxiliary fields. In the next section we will skip off-shell N=2 supersymmetry and move directly to off-shell N=4 supersymmetry which will be of interest to us not for the purposes of manifesting the above properties (even though they will also be apparent there) but because of its expected relation to super- BF theories. The fact that, classically, supersymmetry requires the fermion to vanish is related to the (non-)existence of reducible flat connections as follows. By taking a supersymmetry transformation of the equation of motion for the connection, $F_{ab} = 0$, we find that this implies that the spinors must be gauge-covariantly constant⁵:

$$0 = [Q_\alpha, F_{ab}] \longrightarrow \mathcal{D}_a \psi_\alpha = 0 \ , \tag{2.6}$$

after making use of the properties of the D=3 gamma-matrices. Since this condition holds for any α , we see that if any solutions of the equation $\mathcal{D}_a \Phi = 0$ existed (where Φ is some Lorentz scalar which transforms in the adjoint representation of the gauge group), then there would be non-zero solutions for ψ_α . Thus far we have only used the equation of motion for the gauge field and its supersymmetry transformation. Now, the equation of motion for the fermions impose that they vanish. Thus the existence of reducible

⁵A similar result holds for the B_a field and its super-partner.

connections would at least lead to novel classical behavior for the SUSY- BF theory. This argument, assumes that the Lorentz spin-connection is zero. It would be interesting to extrapolate it to the case of arbitrary curved manifolds with attention focused on the existence of global supersymmetries.

2.2 D=4

It appears that the behaviour we saw above is a universal feature of SUSY- BF theories. As further evidence of this we now turn to the four dimensional theory.

In analogy with three dimensions, in four dimensions we introduce a chiral spinor superfield, \mathcal{B}_α along with the spinor superfield-strength, W_α . The action is roughly of the same form as that of $S_{SUSY}[4, 1]$:

$$S_{SUSY}[4, 1] = -\frac{1}{2} \int d^4\sigma d^2\theta \text{Tr}(\mathcal{B}^\alpha W_\alpha) + (\text{c.c.}) . \quad (2.7)$$

Since W_α satisfies the Bianchi Identities $\nabla^\alpha W_\alpha + \bar{\nabla}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} = 0$ and $\bar{\nabla}_{\dot{\alpha}} W_\alpha = 0$, we find the action to be invariant under separate transformations for B_α :

$$\delta B_\alpha = \nabla_\alpha \Lambda , \quad \delta B_\alpha = (\sigma^a)_\alpha{}^{\dot{\beta}} \bar{\nabla}_{\dot{\beta}} \Lambda_a . \quad (2.8)$$

In these expressions, Λ is a chiral, scalar superfield parameter and Λ_a is an anti-chiral superfield: $\nabla_\alpha \Lambda_a = 0$.

Now, the components of B_α are of the same form as before with the exception that

$$\nabla_\alpha B_\beta = i(\sigma^{ab})_{\alpha\beta} B_{ab} + C_{\alpha\beta} b . \quad (2.9)$$

Then, much as in the three dimensional case, we see from the first transformation in eqn. 2.8, that we can set $\beta_\alpha = 0$. However, unlike three dimensions, not all of b can be set equal to zero. (The B_α multiplet is actually the dilaton multiplet of $N = 1$ four dimensional string theory. The part of b that we cannot set to zero corresponds to the would-be dilaton.) The two form symmetry: $\delta B_{ab} = \mathcal{D}_{[a} \xi_{b]}$ follows from the second superfield transformation.

The component lagrangian differs from that of three dimensions in only two respects. First, the field B is now a two form (with attendant $D=4$

Levi-Cevita tensor) and the fermion term is now of the form $\rho^\alpha \psi_\alpha + (c.c.)$. Hence, as before, we expect that this action is actually invariant under a pair of N=1 supersymmetries.

2.3 Path Integral Measure

Thus far, our constructions have been purely classical. We have obtained SUSY- BF actions without derivative terms for fermions. As was discussed above, this means that the classical supersymmetry theory is the same (modulo the presence of reducible connections) as the non-supersymmetric theory. It also appears that the quantum theories will be the same. However, this goes against our prior experience with supersymmetric theories. Up to signs, the bosonic determinants are supposed to be cancelled by those from fermions. That does not occur here. Thus, we propose to modify the measure of the path integral of SUSY- BF so that this feature of supersymmetric theories will be maintained. In this case, it is easily accomplished as follows. Since the ratio of the non-zero mode determinants which arise [1] from the integration over the connection and B fields is equal to that which appears in the inverse R-S torsion valued in the flat connections, A_0 , we simply insert the R-S torsion, $T[A_0]$ as part of the definition of the flat connection part of the measure for SUSY- BF gauge theories: $[dA_0] \rightarrow [dA_0]T[A_0]$. Since the partition function has support only on flat connections, A_0 in $T[A_0]$ may – in turn – be replaced by the general connection, A . Hence we propose that the path integral for N=1 SUSY- BF theories is

$$\mathcal{Z}_{SUSY} = \int [dA][dB][d\rho][d\psi] T[A] e^{-S_{SUSY}[D,1][A,B,\rho,\psi]} , \quad (2.1)$$

where

$$T[A] = \det^{-3/2} \Delta_A^{(0)} \det^{1/2} \Delta_A^{(1)} ,$$

regardless of the dimension of space-time. Here, $\Delta_A^{(k)}$ is the covariant laplacian on k -forms. Since $T[A_0]$ is topological it does not introduce any local degrees of freedom into the theory.

Having made this insertion, we must determine whether or not the supersymmetries are still preserved in eqn. (2.1). To answer this question we first recall [1] that $T[A]$ may be written as the path integral over a certain

set of fields of the exponential of the Gaussian action for these fields in the gauge background given by the connection, A . Then the coupling of A to these fields in the modified action is of the form $Tr(A \cdot J)$; where J is the gauge current for the additional fields. Consequently, we can maintain the supersymmetries by declaring that the new fields, hence J , do not transform under the latter. This means that the supersymmetry transformation of the action which is used to represent the R-S torsion is of the form $Tr(\delta A \cdot J)$. Such a term may be cancelled by re-defining the supersymmetry transformation law of the fermion which does not appear in δA to be proportional to the current. To conclude this section, we summarize it and offer the following road map for the remainder of the paper. In this section, we have seen that the off-shell $N=1$ supersymmetric BF actions are actually invariant under a pair of these supersymmetries. However, these transformations do not form a $N=2$ supersymmetry algebra, off-shell. Next, we will turn to the formulation of $D=3$, $N=4$ super Yang-Mills. Then, in section 4, we will demonstrate that for SUSY- BF theories, $N=4$ supersymmetric actions can be constructed using “reduced” supermultiplets. This is due to the presence of additional local symmetries in these theories which do not have derivative terms for fermions. We then find that it is these theories which can be twisted (via a Hodge decomposition procedure) to super- BF theories.

3 $D=3$, $N=4$ Super Yang-Mills

There are two off-shell $3D$, $N = 4$ vector supermultiplets. This is an example of a phenomenon that was noted a long time ago in supersymmetric theories [6]. Namely it is often the case that for a given set of propagating fields (on-shell theory), there is one or more distinct off-shell theories. These different off-shell representations of the same physical states are called “variant representation” of the supermultiplet. Sometimes, but not always, variant representations are related by a duality transformation.

The most powerfully known consequence of the existence of variant representations is the occurrence of “mirror symmetry” in compactified heterotic string theory [7]. One way to view mirror symmetry is that it maps a particular off-shell supermultiplet into one of its variant representation and

vice-versa. In two dimensional $N = 2$ theories, this is the situation obtained in heterotic string theories that contain the chiral scalar multiplet and its variant (the twisted chiral multiplet)[8]. The fact that we have discovered the existence of a previously unsuspected 3D, $N = 4$ vector supermultiplet suggest the exciting possibility of extending the concept of mirror symmetry to three dimensions!

Let us start with a 4D, $N = 2$ vector supermultiplet and a 4D, $N = 2$ tensor supermultiplet. The off-shell representations of both of these theories have been known for a long time. Working with the usual actions in four dimensions shows that these two supermultiplets both describe the propagation of 4 bosonic degrees of freedom and 4 fermionic degrees of freedom. Now consider a toroidal compactification to three dimensions. This will necessarily split off some of the components of the gauge fields in each supermultiplet into different 3D representations of the $SO(1,2)$ group. Under the dimensional reduction, the 4D vector gauge field “yields” a 3D vector gauge field as well as one scalar. This process has been called “the inverse Higgs phenomenon.” Similarly, the 4D antisymmetric tensor gauge field “yields” a 3D, 2-form gauge field as well as one 3D vector gauge field. In three dimensions, a 3D 2-form gauge field propagates no physical degrees of freedom. By a duality transformation, it can be replaced by an auxiliary scalar. Also, the number of supersymmetries for the models double because an irreducible spinor representation of $SO(1,3)$ contains two irreducible spinor representations of $SO(1,2)$. So the two models that result are 3D, $N = 4$ vector supermultiplet models. Thus, we arrive at the possible existence of two distinct 3D, $N = 4$ vector supermultiplets!

One of the two distinct 3D, $N = 4$ off-shell vector supermultiplets is given by the following commutator algebra.

$$\begin{aligned}
[\nabla_{\alpha i}, \nabla_{\beta j}] &= i4gC_{\alpha\beta}C_{ij} \bar{W}^{\mathcal{I}} t_{\mathcal{I}} \ , \\
[\nabla_{\alpha i}, \bar{\nabla}_{\beta}^j] &= i2\delta_i^j(\gamma^c)_{\alpha\beta}\nabla_c + 2g\delta_i^j C_{\alpha\beta} S^{\mathcal{I}} t_{\mathcal{I}} \ , \\
[\nabla_{\alpha i}, \nabla_b] &= g(\gamma_b)_{\alpha}{}^{\delta} \bar{W}_{\delta i}^{\mathcal{I}} t_{\mathcal{I}} \ , \\
[\nabla_a, \nabla_b] &= igF_{ab}^{\mathcal{I}} t_{\mathcal{I}} \ .
\end{aligned} \tag{3.1}$$

The following equations must be imposed in order to satisfy the Bianchi identities,

$$\nabla_{\alpha i} \bar{W}^{\mathcal{I}} = 0 \ ,$$

$$\begin{aligned}
\bar{\nabla}_\alpha{}^i \bar{W}^{\mathcal{I}} &= C^{ij} \bar{W}_{\alpha j}^{\mathcal{I}} \quad , \\
\nabla_{\alpha i} S^{\mathcal{I}} &= -i \bar{W}_{\alpha i}^{\mathcal{I}} \quad , \\
\nabla_{\alpha i} \bar{W}_{\beta j}^{\mathcal{I}} &= i 2 C_{ij} (\gamma^c)_{\alpha\beta} (\nabla_c \bar{W}^{\mathcal{I}}) + 2g C_{ij} C_{\alpha\beta} [S, \bar{W}]^{\mathcal{I}} \quad , \\
\bar{\nabla}_\alpha{}^i \bar{W}_{\beta j}^{\mathcal{I}} &= i \delta_j^i (\gamma_a)_{\alpha\beta} \left[\frac{1}{2} \epsilon^{abc} F_{bc}^{\mathcal{I}}(A) + i (\nabla^a S^{\mathcal{I}}) \right] + i C_{\alpha\beta} d^{\mathcal{I}i}_j \quad , \\
\nabla_{\alpha i} d^{\mathcal{I}k}_j &= -[2\delta_i^k \delta_j^l - \delta_j^k \delta_i^l] [(\gamma^c)_\alpha{}^\beta (\nabla_c \bar{W}^{\mathcal{I}}_{\beta l}) + g [\bar{W}_{\alpha l}, S]^{\mathcal{I}} \\
&\quad + g C_{lr} [W_\alpha{}^r, \bar{W}]^{\mathcal{I}}] \quad . \tag{3.2}
\end{aligned}$$

The 3D vector multiplet, we have just discussed is the one that “descends” from the 4D, $N = 2$ vector supermultiplet. The component level equations of motion that follow from the usual action for a vector gauge field begin by setting $d^{\mathcal{I}k}_j = 0$. The first spinorial derivative of this restriction implies the Dirac-like equation for the spinor and the second spinorial derivatives imply the Yang-Mills and Klein-Gordon equations for the bosons.

The second 3D, $N = 4$ vector supermultiplet has a covariant derivative whose commutator algebra takes the form,

$$\begin{aligned}
[\nabla_{\alpha i}, \nabla_{\beta j}] &= 0 \quad , \\
[\nabla_{\alpha i}, \bar{\nabla}_{\beta}{}^j] &= i 2 \delta_i^j (\gamma^c)_{\alpha\beta} \nabla_c + 2g C_{\alpha\beta} \varphi^{\mathcal{I}j}_i t_{\mathcal{I}} \quad , \\
[\nabla_{\alpha i}, \nabla_b] &= g (\gamma_b)_\alpha{}^\delta \bar{U}_{\delta i}^{\mathcal{I}} t_{\mathcal{I}} \quad , \\
[\nabla_a, \nabla_b] &= i g F_{ab}^{\mathcal{I}} t_{\mathcal{I}} \quad . \tag{3.3}
\end{aligned}$$

The following equations must be imposed in order to satisfy the Bianchi identities,

$$\begin{aligned}
\varphi^{\mathcal{I}i}_i &= 0 \quad , \quad \varphi^{\mathcal{I}j}_i - (\varphi^{\mathcal{I}i}_j)^* = 0 \quad , \\
\nabla_{\alpha i} \varphi^{\mathcal{I}k}_j &= -i [2\delta_i^k \bar{U}_{\alpha j}^{\mathcal{I}} - \delta_j^k \bar{U}_{\alpha i}^{\mathcal{I}}] \quad , \\
\nabla_{\alpha i} \bar{U}_{\beta j}^{\mathcal{I}} &= -2 C_{\alpha\beta} C_{ij} \bar{J}^{\mathcal{I}} \quad , \\
\nabla_{\alpha i} \bar{J}^{\mathcal{I}} &= 0 \quad , \quad a^{\mathcal{I}} - (a^{\mathcal{I}})^* = 0 \quad , \\
\bar{\nabla}_\alpha{}^i \bar{U}_{\beta j}^{\mathcal{I}} &= i (\gamma_a)_{\alpha\beta} \left[\frac{1}{2} \delta_j^i \epsilon^{abc} F_{bc}^{\mathcal{I}}(B) + i (\nabla^a \varphi^{\mathcal{I}i}_j) \right] \\
&\quad - i C_{\alpha\beta} [a^{\mathcal{I}} \delta_j^i - \frac{1}{2} g [\varphi_j^k, \varphi_k^i]^{\mathcal{I}}] \quad .
\end{aligned}$$

$$\begin{aligned}
\bar{\nabla}_\alpha^i \bar{J}^{\mathcal{I}} &= iC^{ij} \left\{ (\gamma^a)_\alpha{}^\beta (\nabla_a \bar{U}_{\beta j}^{\mathcal{I}}) - ig [\varphi_j^k, \bar{U}_{\alpha k}]^{\mathcal{I}} \right\} \quad , \\
\nabla_{\alpha i} a^{\mathcal{I}} &= \left\{ (\gamma^a)_\alpha{}^\beta (\nabla_a \bar{U}_{\beta i}^{\mathcal{I}}) - ig [\varphi_i^j, \bar{U}_{\alpha j}]^{\mathcal{I}} \right\} \quad .
\end{aligned} \tag{3.4}$$

This 3D, $N = 4$ vector multiplet is the one that “descends” from the 4D, $N = 2$, 2-form supermultiplet. The component level equations of motion that follow from the usual action for a vector gauge field begin by setting $a^{\mathcal{I}} = \bar{J}^{\mathcal{I}} = 0$. The first spinorial derivative of these restrictions imply the Dirac-like equation for the spinor and the second spinorial derivatives imply the Yang-Mills and Klein-Gordon equations for the bosons.

We should mention that it is actually only the Abelian version of this theory that is obtained by dimensional reduction. The process of replacing the 3D, 2-form by a scalar using a duality transformation is purely a 3D concept. It is crucial to do this for the existence of the non-Abelian version of this theory. It is a highly nontrivial check on the consistency of this second unexpected 3D, $N = 4$ vector supermultiplet that the commutator algebra closes without equations of motion. We have explicitly verified this for the spin-0 and spin-1/2 fields. In this, the following identities are useful

$$\begin{aligned}
\epsilon^{abc} (\gamma_b)_{\alpha\beta} (\gamma_c)_\gamma{}^\delta &= -iC_{\alpha\gamma} (\gamma^a)_\beta{}^\delta - i(\gamma^a)_{\alpha\gamma} \delta_\beta{}^\delta \quad , \\
[\varphi_i^j, \varphi_k^l] &= \frac{1}{2} \delta_k^j [\varphi_i^r, \varphi_r^l] - \frac{1}{2} \delta_i^l [\varphi_k^r, \varphi_r^j] \quad , \\
[\varphi_i^j, \bar{U}_{\alpha k}] &= [\varphi_k^j, \bar{U}_{\alpha i}] + \delta_k^j [\varphi_i^l, \bar{U}_{\alpha l}] - \delta_i^j [\varphi_k^l, \bar{U}_{\alpha l}] \quad .
\end{aligned} \tag{3.5}$$

It turns out that these two distinct supermultiplets are dual to each other in a sense. Notice that the following sets of equations are valid for the physical fields of the first vector supermultiplet and the auxiliary fields of the second vector supermultiplet,

$$\begin{aligned}
\nabla_{\alpha i} \bar{W}^{\mathcal{I}} &= 0 \quad , \quad \bar{\nabla}_\alpha^i \bar{W}^{\mathcal{I}} = iC^{ij} \nabla_{\alpha j} S^{\mathcal{I}} \quad , \\
\nabla_{\alpha i} \bar{J}^{\mathcal{I}} &= 0 \quad , \quad \bar{\nabla}_\alpha^i \bar{J}^{\mathcal{I}} = iC^{ij} \nabla_{\alpha j} a^{\mathcal{I}} \quad .
\end{aligned} \tag{3.6}$$

They “inherit” this duality from their 4D, $N = 2$ vector and 2-form ancestors. This sense of duality is a supersymmetric generalization of Hodge duality that relates a p -form in D dimensions to a $(D - p - 1)$ form and plays a role of utmost importance in formulating an off-shell supersymmetrically consistent 3D, $N = 4$ BF-theory. An explicit calculation shows that the action given by

$$\mathcal{S}_{BF}^{N=4} = \int d^3\sigma \left[\frac{1}{2} \epsilon^{\alpha\beta\gamma} B_a F_{bc}(A) + (W^{\alpha i} \bar{U}_{\alpha i} + U^{\alpha i} \bar{W}_{\alpha i}) \right]$$

$$-Sa - (W\bar{J} + J\bar{W}) - \frac{1}{2}\varphi_i^j d_j^i] \quad , \quad (3.7)$$

(where these are component fields) is invariant under the supersymmetry transformation laws implied by the solution to the Bianchi identities given above for each multiplet.

Earlier, an investigation was undertaken [9] in order to study the role of supersymmetry in Chern-Simons theories. In this previous work, there appeared to be a barrier to finding an off-shell 3D, $N = 4$ Chern-Simons theory. The action just presented above has $N = 4$ supersymmetry, but there also appear two gauge fields in the action. One of the nice features of an off-shell action is that it can be coupled to other superfields. In particular, the gauge fields in the action above may be coupled to 3D, $N = 4$ off-shell matter scalar supermultiplets⁶. On the other hand, the 3D, $N = 4$ CS action⁷ presented in [9] was an on-shell action! In other words it is not possible to couple it to 3D, $N = 4$ matter supermultiplets. There is a very close relation between BF theories and CS theories. In fact, the reason that the BF action above exists as a consistent off-shell theory is because the two different vector supermultiplets used are dual to each other. This should come as no surprise. After all, in 4D ordinary vector gauge fields and 2-form gauge fields are dual to each other. The 4D, $N = 2$ vector and 2-form supermultiplets share this property. Looked at in this way, the solution to the problem of finding a 3D, $N = 4$ CS action is obvious. One must find a 3D, $N = 4$ vector supermultiplet that is self-dual!

Carrying out this search for an off-shell self-dual 3D, $N = 4$ theory turns out to be difficult. To date we have found no solution. So in the following, we briefly report the status of this problem. It is useful to look back at how the two previous 3D, $N = 4$ vector supermultiplets differ. Comparing the two different, commutator algebras, one is first struck by the fact that the two spin-0 degrees of freedom represented by \bar{W} in the commutator algebra have “moved.” They no longer appear in the commutator algebra of $[\nabla_{\alpha i}, \nabla_{\beta j}]$. Instead they re-appear in the commutator algebra of $[\nabla_{\alpha i}, \bar{\nabla}_{\beta}^j]$ in the second version of the supermultiplet. Now what do we mean by the first two vector supermultiplets are “dual” to each other? Well, if one looks at the action, one notices that the “physical fields” of one supermultiplet are in the same $SU(2)$ representation as the “auxiliary fields” of the other supermultiplet and vice-

⁶See our final appendix for a discussion of such matter supermultiplets.

⁷The action given is actually a mixed theory with two CS actions and one BF action.

versa. So a “self-dual” theory would be one in which the physical fields and the auxiliary fields both occur in the same $SU(2)$ representation. There are two ways of doing this. Either the physical fields and auxiliary fields are both $SU(2)$ complex singlets or both $SU(2)$ triplets. The calculations performed so far suggest that in either case this leads to the necessity of adding further auxiliary fields.

In the next section, we will show that the off-shell supersymmetric 3D, $N = 4$ BF actions has some very interesting properties in regard to topological field theory!

4 Reduced Supermultiplets and Twisting

It is generally believed [1] that large classes of TQFT’s may be obtained by twisting (re-defining the Lorentz representations or generators) extended supersymmetric field theories. In two and four dimensions, $N=2$ supersymmetric theories may be twisted [10, 11, 12] to yield TQFT’s in those dimensions. In three dimensions, $N=4$ supersymmetric theories are required [13]. Now, super- BF gauge theories are examples of TQFT’s and they may be constructed via BRST gauge fixing [1]. Thus we know of their existence and form. We now ask how to obtain them via twisting. In this section we will focus on three dimensions. However, we expect that our results are equally applicable in any dimension.

The natural starting point for twisting are the supermultiplets we constructed in the previous section. However, such an attempt immediately leads to two problems. First, we see that the action (3.7) only contains mass terms for the superpartners, whereas it is known that super- BF has derivative terms for these fields. Secondly, under twisting, the last (bosonic) term in $S_{BF}^{N=4}$ leads to a mass term for a pair of vector fields which do not exist in the super- BF theory. Thus, it seems that we need another action from which to start the twisting procedure. In this section, we will solve the second of these problems. The first will be solved in the next section. Normally, as was done in the previous section, the construction of supermultiplets begins by matching the bosonic and fermionic degrees of freedom (*dof*) without reference to an action. This procedure can be misleading as we now point out.

First, the number of bosonic *dof* which appear in the BF action is four, $dof[B] = 2$, $dof[A] = 2$, after accounting for gauge symmetries. Suppose we tried to supersymmetrize this system by introducing a complex, $SU(2)$ doublet, spin- $\frac{1}{2}$ field. Since the number of fermionic *dof* of this field is eight, it would appear that we must also add an additional four bosonic degrees of freedom. However, we should express greater care. The BF action naively had six *dof* which we reduced to four due to its symmetries. Thus we learn that it is important to ascertain the symmetries of the action for the fermionic fields we have introduced. From the results of the previous section, we have seen that their action is that of a mass term without any derivatives. Hence, their action will be invariant under both *local* $SU(2)$ rotations and *local* $U(1)$ transformations. The total number of gauge parameters included in these is four. Thus, for a complex, $SU(2)$ doublet, spin- $\frac{1}{2}$ field, $\Upsilon_{\alpha i}$, whose action is a mass term, the number of degrees of freedom is four, matching the bosonic content of the BF action. Thus, we write the action:

$$S_{BF,red}^{N=4} = \int d^3\sigma \text{Tr} \left[\frac{1}{2} \epsilon^{abc} B_a F_{bc} + \Upsilon^{\alpha i} \bar{\Upsilon}_{\alpha i} \right] . \quad (4.1)$$

It is invariant under the set of rigid transformations

$$\begin{aligned} [Q_{\alpha i}, A_a] &= (\gamma_a)_\alpha{}^\beta \Upsilon_{\beta i} , & \{Q_{\alpha i}, \bar{\Upsilon}_{\beta j}\} &= (\gamma_a)_{\alpha\beta} \epsilon^{abc} \mathcal{D}_b B_a C_{ij} , \\ [\bar{Q}_{\alpha i}, B_a] &= (\gamma_a)_\alpha{}^\beta \bar{\Upsilon}_{\beta i} , & \{\bar{Q}_{\alpha i}, \Upsilon_{\beta j}\} &= -\frac{1}{2} (\gamma_c)_{\alpha\beta} \epsilon^{abc} F_{ab} C_{ij} . \end{aligned} \quad (4.2)$$

As we will see next, it is important that this spin- $\frac{1}{2}$ is charge is a $SU(2)$ doublet.

Unlike the SUSY- BF theories, super- BF theories exist on curved manifolds. This is because under twisting one of the spinor supersymmetry charges becomes as Lorentz scalar. It is only this (identified as the BRST charge) and any other scalar super-charges which must be maintained by the twisting process. Thus, all of the symmetry transformations above, will not be needed in order to obtain the SUSY- BF theory. As the twisting of D=3, N=4 supersymmetric theories has been performed before [13] we will simply state the results of the operation for this simple case. The twisted action is

$$S^T = \int d^3\sigma \text{Tr} \left[\frac{1}{2} \epsilon^{abc} (B_a F_{bc} - \Sigma_{ab} \psi_c) \right] . \quad (4.3)$$

Under twisting, the first term has not changed. The $SU(2)$ fermions have been drastically altered, however. First, they were replaced by bi-spinors

which were then written as a one-form (ψ), a two-form (Σ) and a pair of zero-forms. Then two of the the local gauge symmetry parameters were used to remove the zero-forms. The remaining two local symmetries are realized, in eqn. (4.3) as $\delta\Sigma = \xi^*\psi$ and $\delta\psi = \xi'\Sigma$ for two arbitrary local parameters ξ and ξ' .

The action (4.3) was proposed by us [14] as a cohomological theory for ordinary BF theories. In that work we observed that this action can be written as the anti-commutator of a Grassmann-odd, scalar charge with a certain expression. Now we see how it is connected to (3.7) and (4.1).

5 Twisting Via Hodge Decomposition

We would now like to see how to obtain the SUSY- BF gauge theory from the action S^T . Such a construction requires a new addition to the twisting procedure; namely, Hodge decomposition in a flat connection background. In this section, we will work on arbitrary closed, orientable 3-manifolds, M , with metric. It is then convenient to re-write S^T in terms of forms as

$$S^T = \int_M Tr(B \wedge F - \Sigma \wedge \psi) . \quad (5.1)$$

Furthermore, for convenience, we label the set of fields $\{B, A, \Sigma, \psi\}$ as $\widehat{\mathbf{X}}$ so that $S^T = S^T[\widehat{\mathbf{X}}]$. This action is invariant [14] under the set of symmetries

$$\begin{aligned} [\mathcal{Q}^H, B] &= \psi , & \{\mathcal{Q}^H, \Sigma\} &= F , \\ [Q^H, A] &= \psi , & \{Q^H, \Sigma\} &= \mathcal{D}B , \end{aligned} \quad (5.2)$$

where \mathcal{D} is the covariant exterior derivative.

5.1 Abelian Theory

To set the stage let us first focus attention on the abelian theory. The only field which is a non-singlet under the $U(1)$ group is the connection, A . The Grassmann-odd, two-form, Σ , may be Hodge decomposed as

$$\Sigma \equiv d\chi + {}^*d\eta + \overset{\circ}{\Sigma} \ , \quad (5.3)$$

where χ , η and $\overset{\circ}{\Sigma}$ are Grassmann-odd, 1-, 0- and harmonic 2- forms, respectively, d is the exterior derivative on M and * is the Hodge dual operator. An important point is the absence of χ and η zero-modes in this decomposition; we will return to this point below. Using this eqn. (5.3) in the action (5.1), we find that its partition function is⁸

$$\mathcal{Z} = \int [dB][dA][d\psi][d\chi][d\eta][d\overset{\circ}{\Sigma}] \mathcal{J} \exp - \left\{ \int_M \left[BF - \chi d\psi + \eta d^*\psi + \overset{\circ}{\Sigma} \psi \right] \right\} \ , \quad (5.4)$$

where \mathcal{J} is the Jacobian for the change of variables from the set $\widehat{\mathbf{X}}$ to its Hodge decomposition, $\widehat{\mathbf{Y}}$, in which Σ is replaced by the triplet $(\chi, \eta, \overset{\circ}{\Sigma})$: $\widehat{\mathbf{Y}} = \{\mathbf{B}, \mathbf{A}, \psi, \chi, \eta, \overset{\circ}{\Sigma}\}$. In the partition function for the action $S^T[\widehat{\mathbf{X}}]$, the functional integral over the Σ and ψ fields is equal to one. Thus we determine the Jacobian by requiring that the functional integral over the Grassmann odd fields in (5.4) is also one. That is,

$$\mathcal{Z}_{\Sigma\psi} = \int \left[d\chi, d\psi, d\eta, d\overset{\circ}{\Sigma} \right] \mathcal{J} \exp \left\{ \int_M \left[\chi d\psi - \eta d^*\psi - \overset{\circ}{\Sigma} \psi \right] \right\} \equiv 1 \ . \quad (5.5)$$

At this stage we must be careful about the measures over these fermionic fields. As mentioned previously, in the right-hand-side of equation (5.3), only those fields which are not in $\ker d$, appear. In otherwords, there are χ and η zero-modes present in $\widehat{\mathbf{Y}}$. Yet, when we write the functional integral $\int [d\chi]$, etc., we allow for χ to take values also in $\ker d$. This means that we must define the measures $[d\chi]$ and $[d\eta]$ so that they are equivalent to functional integrals over the non-zero mode parts of the respective fields. We do this by noting that since these fermionic zero-modes are absent from the action, we can simply get rid of the functional integral over them by inserting

⁸Wedge products are understood in the equations to follow.

a complete set in the partition function. Consequently, henceforth, by $[d\chi]$ and $[d\eta]$, we will mean that the respective zero-modes have been inserted in the partition function. Notice that we do not do the same for ψ as its zero-modes explicitly appear in the action in (5.4). Elaboration on this may be found below. The action in the partition function is invariant under the 1-form symmetry. After gauge fixing the latter, we find that

$$\mathcal{Z}_{\Sigma\psi} = b_{(1)} T \det[\Delta_{(0)}] , \quad (5.6)$$

where $b_{(1)} = \dim(H^{(1)}(M))$ (if $\dim H^{(1)}(M) = 0$ then $b_{(1)} = 1$) and T is the R-S torsion on the three dimensional manifold, M . The factor of $b_{(1)}$ arises from the integral over the harmonic 2-form, $\overset{\circ}{\Sigma}$. Being Grassmann-odd, it pulls down that part of ψ which is in $H^{(1)}(M)$. These are Grassmann-odd zero-modes of the exterior derivative on 1-forms: $\psi_{(0)}^I$, $I = 1, \dots, b_{(1)}$. Our ansatz for \mathcal{J} is then

$$\mathcal{J} = \frac{1}{b_{(1)}} T^{-1}(3) \det^{-1} [\Delta_{(0)}] . \quad (5.7)$$

We notice that the last factor can be represented as the Gaussian functional integral of two scalars, λ and ϕ . Putting this together with eqn. (5.4) leads to

$$\begin{aligned} \mathcal{Z} &= T^{-1}(3) \int [dA, dB, d\psi, d\chi, d\eta, d\lambda, d\phi, d\eta', d\lambda', d\phi'] \prod_{I=1}^{b_{(1)}} \psi_{(0)}^I e^{-S_{SBF}} , \\ S_{SBF} &= \int_M [BF - \chi d\psi + \eta d^*\psi + \lambda \Delta\phi + \eta' d^*\chi + \lambda' \Delta\phi'] , \end{aligned} \quad (5.8)$$

where the pair λ' and ϕ' are the anti-ghost and ghost for the gauge-fixing of the 1-form symmetry on χ . S_{SBF} is the action for the first-order (super-BF) form of the Donaldson-Witten theory of flat $U(1)$ connections on three manifolds. The global symmetries mentioned above are manifested in this action as

$$\begin{aligned} [\mathcal{Q}^H, B] &= \psi , & [\mathcal{Q}^H, \chi] &= -A , \\ [Q^H, A] &= \psi , & [Q^H, \chi] &= -B . \end{aligned} \quad (5.9)$$

As before, they are nilpotent.

5.2 Non-Abelian Theory

The preceding discussion carries over to the non-abelian theory with a few twists. One will be the appearance of potential (Yukawa-like coupling) terms in the action. The other has to do with the Hodge decomposition of $ad(G)$ valued forms on M . We address the latter first. Writing Σ as

$$\Sigma \equiv \mathcal{D}\chi + \star \mathcal{D}\eta + \overset{\circ}{\Sigma} , \quad (5.10)$$

is not an orthogonal decomposition, in general. However, it satisfies the latter criteria if the connection is flat. Since our partition function has support only on flat connections, (5.10) is valid in this context. Equations (5.4-5.5) also hold here except that the exterior derivative is replaced by \mathcal{D} and there are traces over the bracketed terms in the action. Having decomposed Σ , we must now determine the transformations on the new fields. In order to do this, it is best to first include the Yang-Mills transformation laws in the action of one of the fermionic generators. By doing this, we identify that the action is invariant under the transformations generated by \mathcal{Q} and $Q \equiv Q^H + Q_{YM}$:

$$\begin{aligned} [\mathcal{Q}, B] &= \psi , & [\mathcal{Q}, \Sigma] &= F , \\ [Q, A] &= \psi - \mathcal{D}\Theta , & [Q, \Sigma] &= \mathcal{D}B + [\Theta, \Sigma] , \\ [Q, B] &= [\Theta, B] , & [Q, \psi] &= [\Theta, \psi] . \end{aligned} \quad (5.11)$$

The commutators on the right-hand-sides of these expressions are those of the gauge algebra. From these we read off that the action of the charges on the new fields are

$$\begin{aligned} [\mathcal{Q}, \overset{\circ}{\Sigma}] &= -F , \\ [Q, \chi] &= [\Theta, \chi] - B , & [Q, \eta] &= [\Theta, \eta] , \\ [Q, \overset{\circ}{\Sigma}] &= [\Theta, \overset{\circ}{\Sigma}] - [\psi, \chi] + [\star\psi, \eta] , \end{aligned} \quad (5.12)$$

and the “horizontal” charges act as

$$[Q^H, A] = \psi , \quad [Q^H, \chi] = -B , \quad [Q^H, \overset{\circ}{\Sigma}] = [\star\psi, \eta] - [\psi, \chi] . \quad (5.13)$$

As in the abelian case, there is a local symmetry manifest in the Hodge decomposition under which we can shift χ by a covariantly exact quantity. We will fix this symmetry later.

The action obtained by combining (5.10) and (5.1) is

$$S^T[\widehat{\mathbf{Y}}] = \int_{\mathbf{M}} \text{Tr} \left(\mathbf{B}\mathbf{F} - \chi \mathcal{D}\psi + \eta \mathcal{D}^*\psi + \overset{\circ}{\Sigma}\psi \right) . \quad (5.14)$$

If we integrate over the harmonic form $\overset{\circ}{\Sigma}$, we find two things. First, the effect of performing the Grassmann-odd integral is to pull down from the action, and into the measure, the zero-modes of ψ which are elements of $H^1(M, G)$. Given the presence of fermionic zero-modes, we might expect the symmetries to be anomalous. This is indeed the case, as the remaining action given by all but the last term in (5.14) is not invariant under the transformations (5.13). Our situation is reminiscent of deriving a component level supersymmetric action by bootstrapping our way term by term. A simple counting of off-shell degrees of freedom shows that we have one more ψ degree of freedom than we have in A ; a similar discrepancy holds between the (χ, η) fields and B . Thus in order to match the number of fermionic and bosonic degrees of freedom, we must add a boson to each of the (B, χ, η) and (A, ψ) supermultiplets. Let us call these λ and ϕ , respectively, so that we have the new supermultiplets⁹ (A, ψ, ϕ) and (B, χ, η, λ) . Then, in order to make the action invariant under (5.13) we must add to it two terms, $\lambda \mathcal{D}^*\mathcal{D}\phi$ and $\lambda[\psi, \star\psi]$, and enlarge the symmetry transformations to include $[Q^H, \lambda] = -\eta$ and $[Q^H, \eta] = [\phi, \lambda]$. With these new terms in the action, we have deduced part of the Jacobian. So we write $\mathcal{J} = \mathcal{J}' \int [d\lambda][d\phi] \exp \{-\int_M \lambda \mathcal{D}^*\mathcal{D}\phi + \lambda[\psi, \star\psi]\}$. It now remains for us to determine \mathcal{J}' . Although we had in mind that the partition function has support only on flat connections when expanding Σ , we can consider the connection in the action to be arbitrary. In this case, the latter is invariant under the local symmetry $\delta\chi = -\mathcal{D}\Lambda$, $\delta B = [\Lambda, \psi]$. Clearly, this is a symmetry of the action if A is a flat connection also. In any case, it must be fixed which we do by selecting the gauge slice $\mathcal{D}^*\chi = 0$. This leads to the action

$$\begin{aligned} S_{SBF} = \int_M \text{Tr} \Big(& BF - \chi \mathcal{D}\psi \\ & + \eta \mathcal{D}^*\psi + \lambda \mathcal{D}^*\mathcal{D}\phi + \lambda[\psi, \star\psi] \\ & + \eta' \mathcal{D}^*\chi + \lambda' \mathcal{D}^*\mathcal{D}\phi' + \lambda'[\psi, \star\chi] \Big) , \end{aligned} \quad (5.15)$$

⁹The A supermultiplet will eventually be enlarged to include an additional Grassmann-odd scalar.

where (as in the abelian case) η' is the gauge-fixing Lagrange multiplier and $(\lambda')\phi'$ is the corresponding (anti-) ghost¹⁰. It is important to note that we are justified in replacing the flat connection in the covariant derivative by A as the path integral has support only on flat connections. We can determine the remaining factor in the Jacobian, \mathcal{J}' . Recall that we must impose

$$\begin{aligned}\mathcal{Z}_{\Sigma\psi} &= \int [d\Sigma][d\psi] e^{-\int_M \text{Tr}(\Sigma\wedge\psi)} \\ &= b^{(1)}(G) \int [d\chi][d\eta][d\psi][d\eta'][d\lambda][d\phi][d\lambda'][d\phi'] e^{-S_{SBF}} \mathcal{J} \quad . \quad (5.16)\end{aligned}$$

The bottom line of this expression may be evaluated if we assume that there are no non-trivial zero-modes of the scalar laplacian, $\Delta_A^{(0)}$. The integrals over ϕ and ϕ' lead to $\delta(\Delta_A\lambda)$ and $\delta(\Delta_A\lambda')$ respectively. With the no zero-mode assumption, these set λ and λ' to be zero, thereby removing the Yukawa couplings. The rest of the integrals are then Gaussians. After diagonalizing them, we find $\mathcal{Z}_{\Sigma\psi} = b^{(1)}(G)\mathcal{J}T(A)$ where $T(A)$ is the R-S torsion¹¹ and $b_{(1)}(G) = \dim(H^1(M, G))$. Hence we have finally found the sought after new representation of the twisted SUSY- BF partition function:

$$\begin{aligned}\mathcal{Z} &= \frac{1}{b^{(1)}(G)} \int [dA][dB][d\chi][d\eta][d\psi][d\eta'][d\lambda][d\phi][d\lambda'][d\phi'] \prod_{I=1}^{b^{(1)}(G)} \psi_{(0)}^I \times \\ &\quad \times T^{-1}(A) e^{-S_{SBF}} \quad . \quad (5.17)\end{aligned}$$

We recognize this as the partition function for the super- BF theory with the ratio of determinants which appears in the inverse R-S torsion (but with the flat connections replaced by the general connection, A) inserted. In addition, the ψ zero-modes appear naturally inserted. The form-degrees, Grassmann-parity and corresponding ghost numbers of the fields which we have introduced in order to write this partition function are given Table I.

¹⁰The supermultiplets are now $(A, \psi, \eta', \phi, \lambda')$ and $(B, \chi, \eta, \lambda, \phi')$, with equal numbers of fermionic and bosonic degrees of freedom.

¹¹Strictly speaking, the R-S torsion is obtained from these integrals only after the B integral is performed as we need the flat connection condition in order to obtain this topological invariant.

FIELD	DEGREE	G-PARITY	GHOST #
B	1	even	0
A	1	even	0
χ	2	odd	-1
ψ	1	odd	1
η	0	odd	-1
η'	0	odd	1
λ	0	even	-2
ϕ	0	even	2
λ'	0	even	0
ϕ'	0	even	0

Collecting the various pieces of the Q^H transformation we find

$$\begin{aligned}
[Q^H, A] &= \psi \ , & [Q^H, \chi] &= B + \mathcal{D}\phi' \ , & [Q^H, \psi] &= \mathcal{D}\phi \ , \\
[Q^H, \lambda] &= -\eta \ , & [Q^H, \eta] &= [\phi, \lambda] \ , & & \\
[Q^H, \lambda'] &= -\eta' \ , & [Q^H, \eta'] &= [\phi, \lambda'] \ . & &
\end{aligned} \tag{5.18}$$

From this we read-off that the square of Q^H is a gauge transformation. The action, S_{SBF} , is known to be Q^H -exact.

6 Concluding Remarks

In this paper, we have constructed the D=3, N=4 super Yang-Mills superspace geometry and used it to construct the corresponding off-shell SUSY- BF gauge theory. We have found that, generically, SUSY- BF gauge theories do not have dynamical fermions; yet, super-BF theories do. In order to twist the 3D, N=4 SUSY- BF theory we found it necessary to Hodge decompose one of the twisted fermions. Additionally, we have seen that in order for the fermion and boson determinants to cancel (up to signs) in the partition function of the SUSY- BF theories, we should re-define the measure to include the Ray-Singer analytic torsion but with flat connections replaced by the full (quantum) connection. After twisting and then Hodge decomposing the SUSY- BF action, we found that the partition function of the SUSY- BF

theory becomes that of the super-BF theory but with the same ratio of determinants of covariant laplacians, which appears in the inverse Ray-Singer torsion, inserted in the measure. It would appear from these results that the only difference between BF and super-BF gauge theories is a choice of vacuum.

Appendix

A Off-shell 3D, $N = 4$ Supergravity

The construction of 3D, $N = 4$ supergravity is essentially equivalent to the problem of finding the consistent truncation of 4D, $N = 2$ supergravity to three dimensions. The fact that this latter problem is a long solved one [15] provides a quick and handy technique for resolution of the three dimensional one. Let us go through the logic that leads to our result. In the off-shell 4D, $N = 2$ supergravity theory, the supermultiplet can be viewed as the direct sum of two 4D, $N = 1$ supermultiplets. One of these supermultiplets is the irreducible non-minimal off-shell 4D, $N = 1$ supergravity supermultiplet [16]. The other multiplet is the off-shell 4D, $N = 1$ matter gravitino supermultiplet [17]. Both of these supermultiplets contain 20-20 bosonic and fermionic degrees of freedom. Of these degrees of freedom, there are 4-4 propagating physical degrees of freedom. In 3D, both supergravity and matter gravitino supermultiplets must consist solely of auxiliary degrees of freedom. This tells us that the truncation to 3D must be such that it separates all of the physical degrees of freedom from the 3D, $N = 2$ supergravity and matter gravitino supermultiplets. The physical degrees of freedom in the case of each supermultiplet wind up in separate 3D, $N = 2$ vector supermultiplets. Thus, we conclude that the 3D, $N = 2$ non-minimal supergravity supermultiplet consist of 16-16 bosonic and fermionic degrees of freedom. The same holds true for the 3D, $N = 2$ matter gravitino multiplet. Now we simply argue that the direct sum of the 3D, $N = 2$ supergravity and matter gravitino supermultiplets must correspond to the 3D, $N = 4$ supergravity multiplet. This observation by itself totally determines the spectrum of the theory we want to construct for the 3D, $N = 4$ case. It must consist of 32-32 bosonic and

fermionic degrees of freedom. Since we have argued that this theory can be directly obtained by the dimensional reduction of the 4D, $N = 2$ supergravity theory along with the truncation described in the paragraph above, we even have *a priori* knowledge of the spectrum of the theory. The knowledge of the spectrum is not sufficient. We also need to know the transformation laws of the supermultiplet. This will be subject of a separate report.

The implementation of this construction at the level of actions is simple. We start with the 4D, $N = 4$ supergravity action which we represent as

$$\int d^4x \mathcal{L}_{N=2,SG} = \int d^4x [\mathcal{L}_{phys} + \mathcal{L}_{aux}] \quad (A.1)$$

where \mathcal{L}_{phys} contains the physically propagating degrees of freedom e_a^m , $\psi_a^{\alpha i}$, $\bar{\psi}_a^{\dot{\alpha} i}$, B_a and \mathcal{L}_{aux} contains all of the auxiliary fields. Under a toroidal compactification, these fields “split” according to the following table.

4D	3D	
SG field	SG fields	Matter fields
e_a^m	e_a^m	$b_a \equiv e_a^3, \phi \equiv e_3^3$
$\psi_a^{\alpha i}$	$\psi_a^{\alpha i}$	$\varphi^{\alpha i} \equiv \psi_3^{\alpha i}$
B_a		$B_a, b \equiv B_3$

Table II

As explicitly seen, two 3D, $N = 2$ vector multiplets appear as matter fields. This implies that in \mathcal{L}_{aux} , two scalar auxiliary fields are associated with the propagating matter fields in our table. These fields are thus part of the two 3D, $N = 2$ vector multiplets. After truncating out these two vector multiplets we are left with an action

$$\int d^3\sigma \mathcal{L}_{N=4,SG} \quad (A.2)$$

The explicit presentation of these results, as well as generalizations involving 3D, $N = 4$ CS supergravity theory, will be given elsewhere.

B 3D, N = 4 Superconformal Algebra

Below we give the form of the superconformal algebra that is associated with the three dimensional off-shell supergravity theory discussed in our text. For the sake of simplicity, only the non-vanishing graded commutators are listed below.

$$[\mathcal{M}_a, \mathcal{M}_b] = \epsilon_{ab}{}^c \mathcal{M}_c \quad , \quad [\mathcal{M}_a, P_b] = \epsilon_{ab}{}^c P_c \quad , \quad [\mathcal{M}_a, K_b] = \epsilon_{ab}{}^c K_c \quad , \quad (B.1)$$

$$[\mathcal{D}, P_a] = P_a \quad , \quad [\mathcal{D}, K_a] = -K_a \quad , \quad (B.2)$$

$$[P_a, K_b] = \frac{1}{2} \eta_{ab} \mathcal{D} - \frac{1}{2} \epsilon_{ab}{}^c \mathcal{M}_c \quad , \quad (B.3)$$

$$[Q_{\alpha i}, \bar{S}_{\beta}{}^j] = C_{\alpha\beta} [\delta_i{}^j (\mathcal{D} + i\mathcal{Y}) - \mathcal{T}_i{}^j] - i \delta_i{}^j (\gamma^c)_{\alpha\beta} \mathcal{M}_c \quad (B.4)$$

$$[\mathcal{M}_a, Q_{\alpha i}] = i \frac{1}{2} (\gamma_a)_{\alpha}{}^{\beta} Q_{\beta i} \quad , \quad [\mathcal{M}_a, S_{\alpha i}] = i \frac{1}{2} (\gamma_a)_{\alpha}{}^{\beta} S_{\beta i} \quad , \quad (B.5)$$

$$[P_a, S_{\alpha i}] = \frac{1}{2} (\gamma_a)_{\alpha}{}^{\beta} Q_{\beta i} \quad , \quad [K_a, Q_{\alpha i}] = \frac{1}{2} (\gamma_a)_{\alpha}{}^{\beta} S_{\beta i} \quad , \quad (B.6)$$

$$[Q_{\alpha i}, \bar{Q}_{\beta}{}^j] = 2 \delta_i{}^j (\gamma^c)_{\alpha\beta} P_c \quad , \quad [S_{\alpha i}, \bar{S}_{\beta}{}^j] = 2 \delta_i{}^j (\gamma^c)_{\alpha\beta} K_c \quad , \quad (B.7)$$

$$[\mathcal{Y}, Q_{\alpha i}] = i \frac{1}{2} Q_{\alpha i} \quad , \quad [\mathcal{Y}, \bar{Q}_{\alpha}{}^i] = -i \frac{1}{2} \bar{Q}_{\alpha}{}^i \quad , \quad (B.8)$$

$$[\mathcal{Y}, S_{\alpha i}] = -i \frac{1}{2} S_{\alpha i} \quad , \quad [\mathcal{Y}, \bar{S}_{\alpha}{}^i] = i \frac{1}{2} \bar{S}_{\alpha}{}^i \quad , \quad (B.9)$$

$$[\mathcal{D}, Q_{\alpha i}] = \frac{1}{2} Q_{\alpha i} \quad , \quad [\mathcal{D}, S_{\alpha i}] = -\frac{1}{2} S_{\alpha i} \quad , \quad (B.10)$$

$$[\mathcal{T}_i{}^j, Q_{\alpha k}] = \delta_k{}^j Q_{\alpha i} - \frac{1}{2} \delta_i{}^j Q_{\alpha k} \quad , \quad [\mathcal{T}_i{}^j, S_{\alpha k}] = \delta_k{}^j S_{\alpha i} - \frac{1}{2} \delta_i{}^j S_{\alpha k} \quad , \quad (B.11)$$

$$[\mathcal{T}_i{}^j, \mathcal{T}_k{}^l] = \delta_k{}^j \mathcal{T}_i{}^l - \delta_i{}^l \mathcal{T}_k{}^j \quad . \quad (B.12)$$

C The 3D, $N = 4$ Supersymmetric $U(1)$ Anyonic Model

One of the nice feature of possessing off-shell supersymmetric representations is that they can be added freely to other such models without regard to loss of supersymmetry. In particular, our construction of the 3D, $N = 4$ sBF (supersymmetric BF) action is such that it can easily be coupled to 3D, $N = 4$ scalar multiplets without new difficulties arising. Thus, we are able to extend the constuction of supersymmetric anyonic models to the level of $N = 4$ supersymmetry. Our construction below marks the first time this has been achieved.

To succeed in this effort requires an off-shell 3D, $N = 4$ scalar supermultiplet. Fortunately, such a representation is available after a little bit of thought. The secret to finding this representation is to recall that one off-shell 4D, $N = 2$ scalar supermultiplet¹² is known in the physics literature, the relaxed hypermultiplet [19]. Using the technique of toroidal compactification leads to a 3D, $N = 4$ scalar supermultiplet! Furthermore, the first of our two vector (since it may be regarded as the dimensional reduction of the 4D, $N = 2$ vector supermultiplet) supermultiplets in (3.1) may be freely coupled to the relaxed hypermultiplet. This opens the way for us to couple our supersymmetric BF action to matter and forming a $N = 4$ anyonic type of model.

The superspace action for our $N = 4$ anyonic theory follows immediately from the corresponding 4D, $N = 2$ theory. The total action consists of the super BF action in section in (3.7) added to the following superfield action¹³

$$\int d^3x d^8\theta [(\lambda_\alpha^i \rho^\alpha_i + \psi_\alpha^i \sigma^\alpha_i + L^{ijkl} X_{ijkl}) + \text{c.c.}] \quad (C.1)$$

In this expression the fundamentally unconstrained superfield potentials are ρ^α_i , $\bar{\sigma}_\alpha^i$ and X_{ijkl} . All other superfields associated with the 3D, $N = 4$ relaxed hypermultiplet are expressed in terms of these fundamental fields as

¹²Actually there is the so-called harmonic space formulation of this supermultiplet [18]. But this leads to a model with an infinite set of auxiliary fields.

¹³We have adhered to the conventions of [19] in the names of quantities below. We warn the reader in particular that the symbols λ and ρ denote superfields below and are not related to the component field given the same names in the body of this work.

$$\begin{aligned}
L^{ij} = & \bar{\nabla}^{ij} \nabla^{3k}{}_{\alpha} \rho^{\alpha}{}_k - \nabla^{ij} \bar{\nabla}^3{}_{k\alpha} \bar{\sigma}^{\alpha k} \\
& - \nabla^{ij} \bar{\nabla}^3{}_{k\alpha} \rho^{\alpha}{}_k - \nabla_k^{(i} \bar{\nabla}^{j)k} \bar{\sigma}^{\alpha k} \bar{\nabla}_{\alpha l} \bar{\sigma}^{\alpha l} \\
& - i \frac{1}{2} \bar{\nabla}^{ij} (\bar{W} \nabla^k{}_{\alpha} \rho^{\alpha}{}_k) - i \frac{1}{2} \nabla^{ij} (W \bar{\nabla}_k{}^{\alpha} \bar{\sigma}_{\alpha}{}^k) \quad ,
\end{aligned} \tag{C.2}$$

$$\lambda_{\alpha}{}^i \equiv \nabla_{\alpha j} L^{ij} \quad , \quad \bar{\psi}_{\alpha}{}^i \equiv \nabla_{\alpha j} L^{ij} \quad . \tag{C.3}$$

$$L^{ijkl} = - \frac{2}{5} \nabla^{(ij} \bar{\nabla}^{kl)} (\nabla^m{}_{\alpha} \rho^{\alpha}{}_m + \bar{\nabla}_m{}^{\alpha} \sigma_{\alpha}{}^m) \quad . \tag{C.4}$$

In these expressions $\nabla_{\alpha i}$ is the gauge covariant derivative that appears in (3.1) and \bar{W} is the corresponding field strength.

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